About some important formulas in the theory of trigonometric series.

by Nicolas Kryloff.

In the theory of Fourier's series or rather in the theory of Fourier's constants, there exists the following well known and very important relation:

$$(1) \qquad \frac{1}{\pi}\int_0^{2\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx,$$

$f(x)$ — being a function, limited [1]) and integrable in the interval $(0, 2\pi)$.

This formula (1), called «équation de fermeture» by M. W. Stekloff was generalised by that scientist for many other functions used in analysis and has received various demonstrations from mathematicians, who were occupied with these questions, which are very important in the whole theory of trigonometric series.

In an article, published some years ago[2]) I had occasion to observe that each formula of trigonometric «summation» will lead us to a new demonstration of (1), and to illustrate this affirmation we propose in this notice to give a demonstration of (1), basing it upon the Vallée-Poussin's formula of summation.

In his remarkable memoir [3]) the Belgian mathematician has established the following summation formula:

$$(2) \qquad S_n = \frac{h_n}{2}\int_{-\pi}^{+\pi} f(u)\left[\cos\frac{u-x}{2}\right]^{2n} du = \frac{h_n g_n}{2}\left[\frac{1}{2}\int_{-\pi}^{+\pi} f(u)du + \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)}\left\{\cos kx\int_{-\pi}^{+\pi} f(u)\cos ku du + \sin kx\int_{-\pi}^{+\pi} f(u)\sin kx du\right\}\right]$$

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[1]) $f(x)$ can also be unlimited (Fatou. Acta Math. t. XXX).

[2]) „Записки Горнаго Института“ 1913 г.

[3]) „Sur l'approximation....“ Bulletin de l'Academie royale de Belgique 1908.

which is no other, than the sum of $n+1$ first terms of Fourier's series, respectively multiplied by one numerical factor, constantly diminishing from one term to another [from the value 1 (for the first term) to the value 0 (for that of rank $n+2$)]; to establish this result it is sufficient to observe, that $\frac{h_n g_n}{2}$ has $\frac{1}{\pi}$, as its asymptotical value.

Putting in (2) $f(x)=1$, we obtain:

$$S_n = \frac{h_n g_n}{4}\int_{-\pi}^{+\pi} dx = \frac{h_n g_n \pi}{2} = \frac{h_n}{2}\int_{-\pi}^{+\pi}\left[\cos\frac{u-x}{2}\right]^{2n} dx$$

and therefore:

$$(3)\qquad f(x).\frac{h_n g_n \pi}{2} = \frac{h_n}{2}\int_{-\pi}^{+\pi} f(x)\left[\cos\frac{u-x}{2}\right]^{2n} du$$

also

$$(4)\qquad \left[S_n - f(x).\frac{h_n g_n \pi}{2}\right] = \frac{h_n}{2}\int_{-\pi}^{+\pi}[f(u)-f(x)]\left[\cos\left(\frac{u-x}{2}\right)\right]^{2n} du =$$

$$= \frac{h_n g_n \pi}{2}\left[\left\{\frac{a_0}{2} + \sum \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)}(a_k \cos kx + b_k \sin kx)\right\} - f(x)\right],$$

where a_i, b_i are the Fourier's coefficients.

Denoting by ω the oscillation of function $f(x)$ in the interval $0 \leq k \leq 2\pi$, we receive for two arbitrary values u and x, belonging to it:

$$-\omega \leq f(u) - f(x) \leq \omega;$$

multiplying now each part of the above inequality by $\frac{h_n}{2}\left[\cos\frac{u-x}{2}\right]^{2n}$ and integrating it, we obtain:

$$-\frac{\omega h_n}{2}\int_{-\pi}^{+\pi}\left[\cos\left(\frac{u-x}{2}\right)\right]^{2n} du \leq \frac{h_n}{2}\int_{-\pi}^{+\pi}[f(u)-f(x)]\left[\cos\frac{u-x}{2}\right]^{2n} du \leq$$

$$\leq \frac{\omega . h_n}{2}\int_{-\pi}^{+\pi}\left[\cos\frac{u-x}{2}\right]^{2n} du;$$

hence by help of the relations (3) and (4), we have:

$$-\frac{\omega . h_n g_n \pi}{2} \leq$$

$$\leq \frac{h_n g_n \pi}{2}\left[\left\{\frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{n(n-1)\ldots(n-k+1)}{(n+1)\ldots(n+k)}(a_k \cos kx + b_k \sin kx)\right\} - f(x)\right] \leq$$

$$\leq \frac{\omega . h_n g_n \pi}{2}$$

or what is the same

$$-\omega \leq S'_n - f(x) \leq \omega,$$

where S'_n is the sign of V. Poussin's sum; therefore we are assured, that the absolute value of $S'_n - f(x)$ is not greater than the oscillation of function in the interval $(0, 2\pi)$.

To establish the equality:

$$\lim_{n=\infty} V_n = \lim_{n=\infty} \int_{-\pi}^{+\pi} |S'_n - f(x)|\, dx = 0, \tag{5}$$

we proceed to demonstrate some introductory results: let $\delta = (a, b)$ be one part of interval $(-\pi, +\pi)$, upon which the oscillation of function is equal to k, then for each value of x belonging to this interval

$$\frac{h_n}{2}\int_a^b [f(u) - f(x)] \left[\cos\frac{u-x}{2}\right]^{2n} du \leq \frac{kh_n}{2}\int_a^b \left[\cos\frac{u-x}{2}\right]^{2n} du \leq$$

$$\leq \frac{kh_n}{2}\int_{-\pi}^{+\pi} \left[\cos\frac{u-x}{2}\right]^{n} du = \frac{kh_n g_n \pi}{2} = k,$$

when $\lim n = \infty$; denoting now by $\delta = (a_1, b_1)$—interval, lying inside of (a, b), so that $a < a_1 < b_1 < b$, and by x one point belonging to this interval, we have:

$$\left| S_n - f(x)\frac{h_n g_n \pi}{2} \right| = \frac{h_n}{2}\int_{-\pi}^{a} [f(u) - f(x)] \left[\cos\left(\frac{u-x}{2}\right)\right]^{2n} du + \frac{h_n}{2}\int_a^b + \frac{h_n}{2}\int_b^{\pi}$$

when u lies in intervals $(-\pi, a)$, or (b, π) and x in (a_1, b_1); then $u - x \neq 0$ and $\left|\cos\frac{u-x}{2}\right| = \varepsilon < 1$; therefore

$$\lim_{n=\infty} \frac{h_n}{2}\int_{-\pi}^{a} = 0; \quad \lim_{n=\infty} \frac{h_n}{2}\int_b^{\pi} = 0$$

and the to middle part $\frac{h_n}{2}\int_a^b$ we can apply the observation made above; hence

$$|S'_n - f(x)| < k + \eta,$$

where $\lim_{n=\infty} \eta = 0$ and k is the oscillation of $f(x)$ over interval (a, b).

Return now to formula (5), we remember that because the condition about the integrability of $f(x)$ is given, it will be possible to divide the whole interval $(-\pi, +\pi)$ into such parts, that the sum of those [1]), upon which the oscillation of the function would be greater then σ, can be made less then an arbitrary small quantity ε; upon others, that we denote by δ_1, the oscillation of the function $\leq \sigma$. Then dividing δ_1 into such parts δ_2 and δ_3, where δ_2 would be lying inside the interval δ_1 and that the sum δ_3 would be less, than arbitrary small ε_1, we receive:

$$\int_{-\pi}^{+\pi}|S'_n(x)-f(x)|\,dx = \sum_{\delta_3}\int_{\delta_3}|S'_n(x)-f(x)|\,dx + \sum_{\delta_2}\int_{\delta_2}|S'_n(x)-f(x)|\,dx + \\ + \sum_{\delta}\int_{\delta}|S'_n(x)-f(x)|\,dx,$$

where, for example, $\sum_{\delta_2}\int_{\delta_2}$ denotes the sum of integrals extended over the intervals δ_2.

Now, we observe, that in the middle sum:

$$|S'_n(x)-f(x)| \leq \sigma + \eta,$$

as was established above, and in the other sums evidently:

$$|S'_n(x)-f(x)| \leq \omega;$$

therefore:

$$\int_{-\pi}^{+\pi}|S'_n(x)-f(x)|\,dx \leq \omega\sum\delta_3 + 2\pi(\sigma+\eta) + \omega\sum\delta;$$

but

$$\sum\delta \leq \varepsilon; \quad \sum\delta_3 \leq \varepsilon_1; \quad \lim_{n=\infty}\eta = 0;$$

and σ depends solely on our choice; consequently we have:

$$\lim_{n=\infty}\int_{-\pi}^{+\pi}|S'_n(x)-f(x)|\,dx = 0,$$

which is just the desired result.

This stated, we observe that in the case of limited functions the following relation is true:

$$\left|\int_{-\pi}^{+\pi}|f^2(x)-S'^2_n(x)|\,dx\right| \leq 2L\int_{-\pi}^{+\pi}|S'_n(x)-f(x)|\,dx,$$

where L denotes the maximum of $f(x)$; therefore

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[1]) Let denote them by δ.

$$\int_{-\pi}^{+\pi} f^2(x)\,dx = \lim_{n-\infty} \int S'^2_n(x)\,dx;$$

but

$$\int_{-\pi}^{+\pi} S'^2_n(x)\,dx = \pi\left[\frac{a_0^2}{2} + \sum(a_n^2 + b_n^2)\left(\frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)}\right)^2\right]$$

and since from the inequality of Bessel follows at once the convergence of the series $\sum a_n^2 + b_n^2$; using now the well known property of convergence factor $\left[\frac{n(n-1)\ldots(n-x+1)}{(n+1)(n+2)\ldots(n+x)}\right]^2$, and applying the famous Abel's reasoning, we have:

$$\lim_{n=\infty} \int_{-\pi}^{+\pi} S'^2_n(x)\,dx = \pi\left[\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2)\right]$$

and therefore

$$\frac{1}{\pi}\int_{-\pi}^{+\pi} f^2(x)\,dx = \frac{a_0^2}{2} + \sum_1^\infty a_n^2 + b_n^2$$

i.e the equation of «fermeture», demonstrated here by means of M-r Vallée-Poussin's method of summation.

Observation: in the same manner as above it would be possible to establish the relation: $\lim_{n=\infty} \int_{-\pi}^{+\pi} [S'_n(x) - f(x)]^2 dx = 0$.

In fact using the notation:

$$c_k = c'_k = \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)} = \frac{A_n}{B_n},$$

we have:

$$\int_{-\pi}^{+\pi} [S'_n(x) - f(x)]^2 dx = \int_{-\pi}^{+\pi} [f(x) - \frac{1}{2}c_0 a_0 + \sum_{k=1}^{n} c_k a_k \cos kx + c'_k b_k \sin kx]^2\,dx =$$

$$= \int_{-\pi}^{+\pi} [f^2(x) - \pi\left[\frac{1}{2}c_0(2 - c_0)a_0^2\right] + \sum_{k=1}^{n} [c_k(2 - c_k)a_k^2 + c'_k(2 - c'_k)b_k^2]$$

but

$$c_k(2 - c_k) = \frac{A_n}{B_n}\left(2 - \frac{A_n}{B_n}\right) = \frac{2A_nB_n - A_n^2}{B_n^2},$$

hence by addition and subtraction of

$$\frac{B_n^2}{B_n^2}a_k^2 = a_k^2, \quad \frac{B_n^2}{B_n^2}b_k^2 = b_k^2,$$

we obtain

$$(6)\quad \left\{\int_{-\pi}^{+\pi} f(x)\,dx - \pi\left[\frac{a_0^2}{2} + \sum_{k=1}^{n} a_k^2 + b_k^2\right]\right\} + \left\{\sum\left[\frac{A_n - B_n}{B_n}\right]^2 [a_k^2 + b_k^2]\right\} = \\ = \int_{-\pi}^{+\pi} [S'_n(x) - f(x)]^2\,dx$$

and because

$$\pi\left[\frac{a_0^2}{2} + \sum a_k^2 + b_k^2\right] \leq \int_{-\pi}^{+\pi} f(x)^2\,dx,$$

the two parts of the left side of (6) are positive and consequently by passage to limit, we receive the desired equation ot «fermeture».

Taking this occasion we undertake here briefly to discuss the degree of accuracy with which functions of real variables having simple discontinuity [1]) can be represented by means of M-er Vallée-Poussin's approximating function.

This function by simple change of the variables and because of the periodicity of the functions under the sign of integration, can be represented as follows:

$$P_n(x) = h_n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x + 2u)\left[\cos u\right]^{2n} du,$$

where the asymptotical value of

$$\frac{1}{h_n} = 2\int_0^{\frac{\pi}{2}} \cos u^{2n}\,du$$

is, after M-er Poussin's investigations, equal to $\sqrt{\frac{\pi}{2}}$.

From the preceeding formulas we obtain immediatly:

$$P_n(x) - f(x) = h_n \int_0^{\frac{\pi}{2}} [f(x + 2u) + f(x - 2u) - 2f(x)] \cos^{2n} u\,du$$

the division of the interval in two parts $\left(0, \frac{\delta}{2}\right)$, $\left(\frac{\delta}{2}\ \frac{\pi}{2}\right)$ gives us:

$$P_n(x) - f(x) = h_n\left[\int_0^{\frac{\delta}{2}} + \int_{\frac{\delta}{2}}^{\frac{\pi}{2}}\right] \leq \left[4\lambda \int_0^{\frac{\delta}{2}} u \cos^{2n} u\,du + 2v \int_{\frac{\delta}{2}}^{\frac{\pi}{2}} \cos^{2n} u\,du\right] h_n,$$

[1]) The case of continued function has been already treated by M-er Vallée-Poussin in his above mentionid memoir.

if we assume, that x is the middle of a closed interval of length $2\delta \leq 2\pi$, in which $f(x)$ satisfies the Lipschitz condition

$$|f(x_2) - f(x_1)| \leq \lambda\,|x_2 - x_1|;$$

by v is denoted the oscillation of $f(x)$ in the whole interval of integration; then evidently:

$$|f(x+2u) + f(x-2u) - 2f(x)| \leq |f(x+2u) - f(x)| + |f(x-2u) - f(x)|$$

and, accordingly, the above written relation is justified.

Because of the evident relation, $\cos u < e^{-\frac{u^2}{2}}$, we obtain:

$$(\alpha) \qquad |P_n(x) - f(x)| < 4\lambda . h_n \int_0^{\frac{\delta}{2}} u e^{-nu^2} du + h_n . 2v \int_{\frac{\delta}{2}}^{\frac{\pi}{2}} \cos^{2n} u du;$$

the first integral of the right side is evidently less than

$$\frac{1}{n}\int_0^{\infty} v e^{-v^2}\, dv = \frac{A}{n}, \qquad \text{(where } A = \text{const.)}$$

and the second

$$\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} [\cos^2 u]^n du = \int_{\frac{\delta}{2}}^{\frac{\pi}{2}} (1 - \sin^2 u)^n du$$

by means of the relation $\sin u \gtrless \frac{2u}{\pi}$, holding for the interval $0 \leq u \leq \frac{\pi}{2}$, is evidently less than

$$\int_{\frac{\delta}{2}}^{\frac{\pi}{2}} \left(1 - \frac{4u^2}{\pi^2}\right)^n du = \frac{\pi}{2}\int_{\frac{\delta}{\pi}}^{1} (1 - v^2)^n dv,$$

where $\frac{2u}{\pi} = v$; a new change of variables gives us:

$$\frac{\pi}{2}\int_{\frac{\delta}{\pi}}^{1} (1 - v^2)^n du = \frac{\pi}{4}\int_{\frac{\delta^2}{\pi^2}}^{1} (1-u)^n \frac{du}{\sqrt{u}} \leq \frac{\pi^2}{4\delta}\int_{\frac{\delta^2}{\pi^2}}^{1} (1-u)^n du = \frac{\pi^2}{4\delta}\left[\frac{[1-u]^{n+1}}{(n+1)}\right]_1^{\frac{\delta^2}{\pi^2}} =$$

$$= \frac{\pi^2 \left[1 - \frac{\delta^2}{\pi^2}\right]^{n+1}}{4\delta\,(n+1)} < \frac{\pi^2}{4\delta\,(n+1)}$$

therefore from (α) we obtain:

$$(\beta) \qquad |P_n(x) - f(x)| < \frac{B.\lambda}{\sqrt{n}} + \frac{Cv}{\sqrt{n}.\delta}, \text{ (where } B, C \text{ are the constants)}$$

the preceeding result can be summed up as follows.

Theorem: *Let $f(x)$ be a function of x, of period 2π, finite and integrable; then the approximating function of M. Vallée-Poussin for evay point x, lying in the middle of a closed interval of length $2\delta \leq 2\pi$, in which $f(x)$ satisfies the Lipschiz condition, posseses the property, expressed by the above formula (β), where v is the oscillation of $f(x)$ in the interval* $-\pi \leq x \leq \pi$.

This theorem completes the interesting result of M. Wilder (in his recent memoir)[1] as concerns M. Vallée-Poussin's summation and evidently in the same manner the other theorems of M. Wilder can be extended.

To close, it would be perhaps not without certain interest to observe, that the well known theorem concerning the possibility of integration term by term of the trigonometric series can be demonstrated, basing upon V.-Poussin's formula of summation, as follows: taking the formula

$$S_n = \frac{h_n}{2}\int_{-\pi}^{+\pi} f(x) \left[\cos\frac{u-x}{2}\right]^{2n} du$$

and observing that:

$$S_n = \frac{h_n g_n}{2}\left\{\frac{1}{2}\int_{-\pi}^{+\pi} f(u)\, du + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)}\left[\cos kx \int_{-\pi}^{+\pi} f(x) \cos kudu + \sin kx \int_{-\pi}^{+\pi} f(x) \sin kxdx\right\}\right]$$

we have surely:

$$\int_{-\pi}^{x} S_n(x)\, du = \frac{h_n g_n \pi}{2}\left[a_0 x + \left(a_0\pi - \sum_{k=1}^{n} \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)} \frac{(n-1)^k a_k}{k}\right) + \right.$$

$$\left. + \sum_{k=1}^{n} \frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)}\left(\frac{a_k \sin kx - b_k \cos kx}{k}\right)\right]$$

but, remembering that:

$$\lim_{n=\infty}\int_{-\pi}^{x} S_n(x)dx = \int_{-\pi}^{x} f(x)dx; \quad \lim_{u=\infty}\frac{h_n g_n \pi}{2} = 1; \quad \lim_{n=\infty}\frac{n(n-1)\ldots(n-k+1)}{(n+1)(n+2)\ldots(n+k)} = 1,$$

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1) «On the degree of approximation to discontinuous funetions, etc.» Rendiconti del Circolo mat. di Palermo. t. XXXIX.

we find the required formula

$$(8)\quad \int_{-\pi}^{x} f(x)dx - a_0 x = \left(a_0\pi - a_1 + \frac{a_2}{2} - \ldots\right) + \sum_{1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n},$$

if we can establish the absolute convergence of series:

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots; \quad (9)\ b_1 + \frac{b_2}{2} + \frac{b_3}{3} + \ldots$$

but in consequence of Cauchy's inequality, we have

$$\left(\frac{a_n}{n} + \frac{a_{n+1}}{n+1} + \ldots\right) \leq \sqrt{a_n^2 + a_{n+1}^2 + \ldots} \sqrt{\frac{1}{n^2} + \frac{1}{(n+1)^2} + \ldots},$$

i.e the desired result, because the same reasoning can be repeated for the series (9).

Because the expression of x:

$$2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \ldots\right)$$

is a series that does not converge absolutely, we can say: in order that a function $F(x)$, possessing the finite and integrable derivative $f(x)$, may be developed in an absolutly convergente trigonometric series, it is necessary and sufficient, that $a_0 = \pi \int_{-\pi}^{+\pi} f(x)dx = 0$; this follows imnmediatly from (8) as M. V.-Poussin has remarked in an article [1]) involving the same questions, but from the point of view of Poisson's method of summation.

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Gagry. Caucasus.
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[1]) „Sur q.q. applications de l'intégrale de Poisson". Bull. de l'Ac. royale de Belgique. 1892.